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# The formula for the lower estimate of the fundamental frequency of natural vibrations of a truss with an arbitrary number of panels

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**Abstract:**

***The object of the research*** is a planar, statically determinate girder of the beam type with a triple diagonal lattice.The truss mass is modeled by equal masses distributed over the nodes of the lower chord. By the Dunkerley method, under the assumption of vertical vibrations of loads, a lower analytical estimate of the first natural vibration frequency is obtained. ***Method.*** The forces in the members are calculated by cutting out nodes from the solution of a system of linear algebraic equations. Generalization of individual solutions to the case of an arbitrary number of panels is carried out by the induction method with the involvement of operators of the Maple computer mathematics system. ***Results.*** Comparison with the numerical solution found from the solution on the spectrum of natural vibrations of a multi-mass system shows that the estimation accuracy depends on the number of panels and varies from 16% for trusses with two panels to 4% for trusses with more than 11 panels. With a decrease in the ratio of the panel height to its length, the accuracy slightly increases. Based on the analysis of the derived formula, it is shown that the dependence of the first frequency on the height of the truss has a maximum. An algorithm for generalizing the solution to the case of members of different stiffness is proposed.

## 1 Introduction

The value of the first vibration frequency of the truss, along with such characteristics as rigidity and strength, is one of the most important operational characteristics of the structure. If the system has many degrees of freedom, then the calculation of the natural frequency is possible only analytically. However, in many cases an approximate estimate of the first frequency is sufficient for a designer or researcher. For structures of a regular type, such an estimate, depending on the number of panels, can be obtained analytically. For some planar trusses with a simple lattice, such dependences were obtained in в [1], [2], [3]. The basis for solving the problem of the lower frequency of the truss can be the solution of the problem of its deflection [4]. There are also known formulas for the dependence of the deflection of regular trusses on the number of panels [5]–[10]. For the first time, general questions of statically definable rod (planar and spatial) systems were considered in [11]–[13]. The natural vibrations of planar regular trusses were studied analytically in [14]–[16]. Analysis of the spectra of natural vibrations of building structures is in demand in solving problems of seismic safety [17]–[21] and in optimization problems [19], [22]–[24]. Combined methods are also used to study the vibrations of regular structures [4], [25]. Analytical solutions of the problem of calculating the deflection for 72 plane statically determinate regular trusses are contained in the handbook [26].

# 2 Materials and Methods

## 2.1 Truss scheme. Calculation of the compliance matrix

Consider a beam-type truss with a diagonal lattice. The inertial properties of the truss are modeled by identical weights located at the nodes of the lower belt. A truss with *n* panels in a half span contains $\mu = 16n + 4$ members. Each panel has two ascending braces and one descending brace. The truss is symmetrical. Ignoring horizontal displacements, consider only vertical vibrations of the weights. In this case, the number of degrees of freedom of the system of truss loads is equal to *N* = 6*n* – 1. The length of each panel is *a*, the height is *h*. For trusses of this type, analytical methods are available for finding the dependences of deformations and forces on the number of panels.

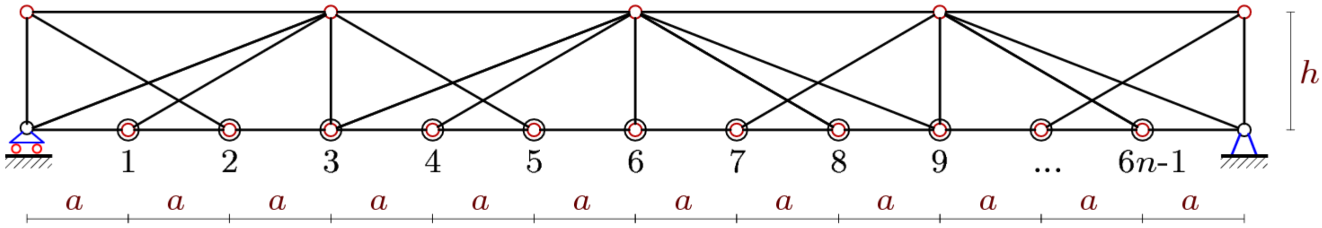


**Fig. 1. Truss, *n*=2**

Differential equations for the dynamics of the system of loads are as follows:

$$\mathbf{M}_N \ddot{\mathbf{Y}} + \mathbf{D}_N \mathbf{Y} = 0\,, \tag{1}$$

where $\mathbf{Y} = [y_1, y_2, ..., y_N]^T$ – vertical displacements of masses, $\mathbf{D}_N$ – stiffness matrix, 2 $\mathbf{M}_N$ –diagonal inertia matrix of size $N \times N$, $\ddot{\mathbf{Y}}$ – acceleration vector. If the masses are the same, then the inertia matrix is expressed through the unit $\mathbf{M}_N = m\mathbf{I}_N$. Compliance matrices $\mathbf{B}_N$ inverse to stiffness matrix $\mathbf{D}_N$. Its elements are determined by the Maxwell-Mohr formula

$$b_{i,j} = \sum_{\alpha=1}^{\mu} S_\alpha^{(i)} S_\alpha^{(j)} l_\alpha / (EF)\,, \tag{2}$$

where $EF$ is the stiffness of the rods, $S_\alpha^{(i)}$ is the force in the member $\alpha$ from the action of a single vertical force at node *i*, $l_\alpha$ is the length of the member $\alpha$. The support rods are assumed to be non-deformable, and the summation in (2) does not apply to these rods. To find efforts, you can use the program in the Maple system [27], which gives efforts in an analytical form. The program uses the method of cutting nodes and solving the joint system of equilibrium equations for all nodes of the truss. The unknowns also include support reactions. The system matrix includes the values of the direction cosines of the forces found from the coordinates of the regular grid of nodes.

Multiplying (1) by the left, taking into account the identity valid for harmonic oscillations of the form

$$y_k = u_k \sin(\omega t + \varphi_0)\,, \tag{3}$$

the problem is reduced to the problem of the eigenvalues of the matrix $\mathbf{B}_N$: $\mathbf{B}_N \mathbf{Y} = \lambda \mathbf{Y}$, where $\lambda = 1/(m\omega^2)$ is the eigenvalue of the matrix $\mathbf{B}_N$, $\omega$ is the natural frequency of oscillations. This problem can only be solved numerically.

## 2.2 Dunkerley's method

Consider an approximate solution according to the Dunkerley method. The lower estimate of the first vibration frequency is given by the formula:

$$\omega_D^{-2} = \sum_{k=1}^{N} \omega_k^{-2}\,, \tag{4}$$

where $\omega_k$ is the vibration frequency of one mass *m* located at the node *k* + 1 of the lower belt, taking the numbering of the nodes from the left support. Equation (1) in the case of oscillations of one mass has a simple scalar form:

$$m\ddot{y}_k + d_k y_k = 0,$$

where $y_k$ is the vertical displacement of the mass, $\ddot{y}_k$ is the acceleration, $d_k$ is the stiffness coefficient (*k* is the mass number). Frequency of vibration of the load is $\omega_k = \sqrt{d_k / m}$. The stiffness coefficient, the reciprocal of the compliance coefficient, is determined by the Maxwell-Mohr formula:

$$\delta_k = 1 / d_k = \sum_{\alpha=1}^{\mu} \left(\tilde{S}_\alpha^{(k)}\right)^2 l_\alpha / (EF) \,.$$

Here it is indicated $\tilde{S}_\alpha^{(k)}$ - the forces in the element numbered $\alpha$ from the action of a unit vertical force applied to the node where the mass numbered *k* is located. According to (4) we have

$$\omega_D^{-2} = m\sum_{k=1}^{N} \delta_k = m\Delta_n. \tag{5}$$

Calculating successively the sums $\Delta_n$, we notice the general form of the solution

$$\Delta_n = (C_{1,n}a^3 + C_{2,n}h^3 + C_{3,n}c^3 + C_{4,n}d^3) / (h^2 EF) \tag{6}$$

and we get a sequence of formulas:

$$\Delta_1 = (171a^3 + 9h^3 + 5c^3 + 9d^3) / (3h^2 EF),$$
$$\Delta_2 = (39618a^3 + 481h^3 + 542c^3 + 504d^3) / (72h^2 EF),$$
$$\Delta_3 = (204120a^3 + 1127h^3 + 1416c^3 + 891d^3) / (81h^2 EF),$$
$$\Delta_4 = (738612a^3 + 2403h^3 + 3020c^3 + 1440d^3) / (96h^2 EF),$$
$$\Delta_5 = (4160475a^3 + 9023h^3 + 11125c^3 + 4275d^3) / (225h^2 EF),...$$

The general terms of the sequences of $a^3,\ h^3,\ c^3,\ d^3$ coefficients at have the form

$$\begin{aligned} C_{1,n} &= (288n^5 + 180n^3 + 80n^2 + 7n + 15) / (10n), \\ C_{2,n} &= (36n^4 - 54n^3 + 97n^2 - 26n + 1) / (18n^2), \\ C_{3,n} &= (36n^3 - 11n + 5) / (18n), \\ C_{4,n} &= 4n - 1. \end{aligned} \tag{7}$$

When finding common terms, the rgf_findrecur operator from the special genfunc package of the Maple system was used, which gives recurrent equations for the elements of sequences, and then the rsolve operator to solve these equations.

## 3 Results and Discussion

Taking into account (5) and (6), we obtain the final formula for the lower boundary of the first natural vibration frequency of the truss:

$$\omega_D = h\sqrt{\frac{EF}{m(C_1a^3 + C_2h^3 + C_3c^3 + C_4d^3)}} \,. \tag{8}$$

We estimate the error of estimate (5) from comparison with the numerical solution of the problem of oscillation of a system with the number of degrees of freedom *N*. The eigenvalues of the matrix $\mathbf{B}_N$ are determined using the operator Eigenvalues from the LinearAlgebra package of the Maple system. The graph (2) compares the curves of the dependence of the first frequency, obtained numerically and according to formula (8). The elastic modulus for steel is adopted: $E = 2\cdot 10^5$ Мпа, the cross-sections of the rods are the same: $F = 40{,}5$ см$^2$. The nodes contain masses *m* = 1500 kg. The following dimensions are taken: *a* = 3 m, *h* = 2 m. The curve constructed by the analytical solution (8) naturally turns out to be lower than the numerical solution. With an increase in the number of panels, the accuracy of the estimate obtained increases. This can be seen in the graph (3) of the dependence of the relative error $\varepsilon = (\omega_1 - \omega_D) / \omega_1$.

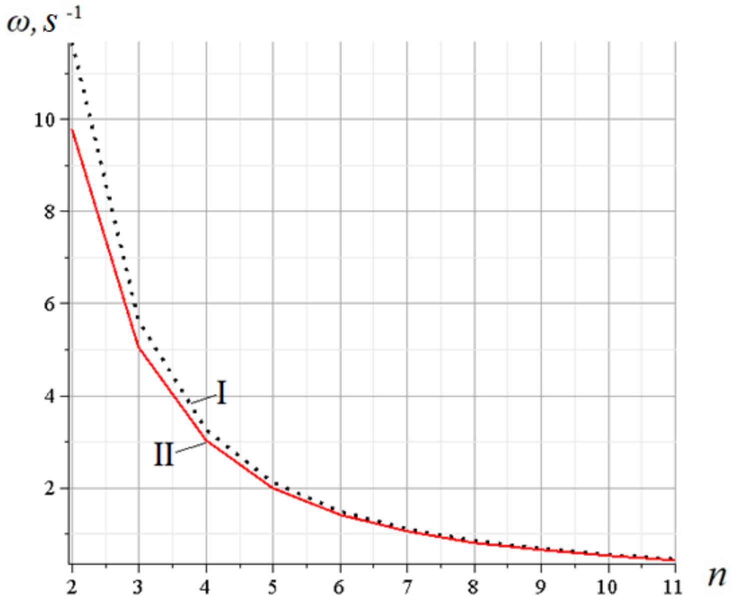


**Fig. 2. Frequency dependence on the number of panels,**
**I – numerical solution; II – analytical assessment**

It is also noticed that the choice of dimensions, stiffness and mass has almost no effect on the accuracy of the lower bound. The number of panels has the greatest impact.

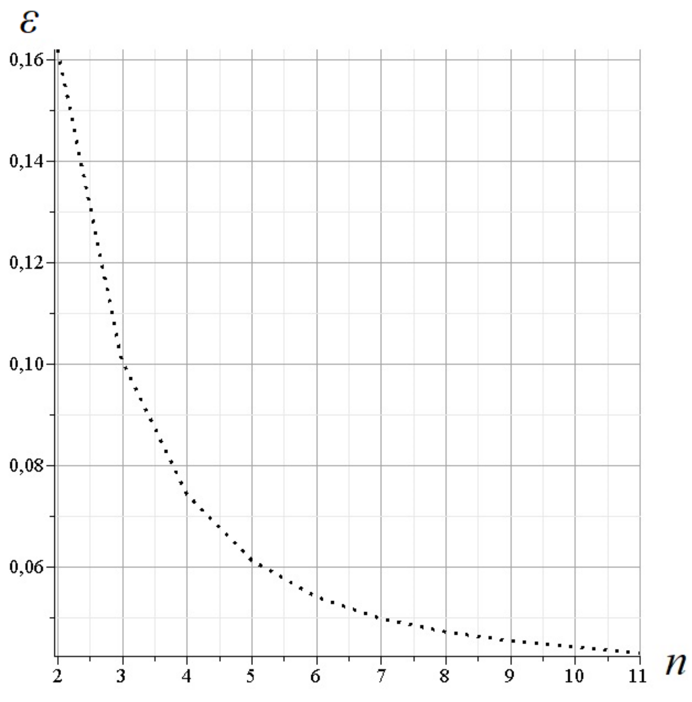


**Fig. 3. Dunkerley's estimation error depending on the number of panels**

The solution error, depending on the number of panels, monotonously changes from 16% at n = 2 to 4%, with a large number of panels.

Calculations using formula (8) show that the vibration frequency depends nonlinearly on the panel height h (Fig. 4). The plots are plotted at $a = L / n$, where $L$ is the span and the same values of the masses and stiffness of the rods as the previous plots. With an increase in the number *n* of panels in the truss, the maximum point shifts to the left on the graph, and the value of the extreme frequency decreases.

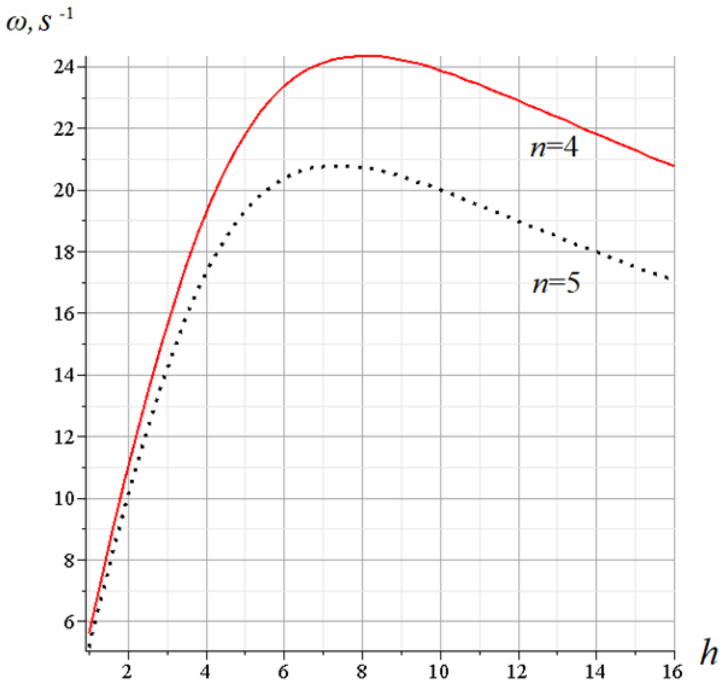


**Fig. 4. Dependence of the vibration frequency on the height of the truss**

Attempts to obtain the upper limit of the lowest frequency in an analytical form using the Rayleigh energy method [28] showed that the final formula turns out to be too cumbersome for practical use. It is known that the Rayleigh method is more accurate than the Dunkerley method.

The solution to the problem for estimating the first frequency, obtained by the induction method, has a closed form, does not contain sums and series, and is not associated with the use of special functions and with iterative calculations. The polynomials in the number of panels included in the final formula have degree no higher than five. The formulas can be used to estimate the oscillation frequency of a truss with a very large number of rods, that is, precisely in those cases where the accumulation of computational errors is most likely and difficulties arise with the amount of computations. The obtained estimate can also be used for a truss with different stiffness of bar elements. For this, without changing functions (7), it is enough to introduce the relative stiffness coefficients of rods of different lengths: $EF_a = \gamma_a EF,\ EF_h = \gamma_h EF,\ EF_c = \gamma_c EF\ , EF_d = \gamma_d EF$ , .. Formula (6) in this case takes the form

$$\Delta_n = (C_{1,n}a^3 / \gamma_a + C_{2,n}h^3 / \gamma_h + C_{3,n}c^3 / \gamma_c + C_{4,n}h^3 / \gamma_d) / (h^2 EF).$$

## 4 Conclusions

Main results of the work are as follows.

1. A non-standard scheme of the truss lattice is proposed, which allows an analytical solution for the forces in the rods.

2. An explicit dependence of the main frequency of oscillations of the truss on its size, mass, and number of panels has been obtained.

3. The high accuracy of the found analytical solution is shown with a large number of panels.

4. An extremum of the dependence of the fundamental frequency on the height of the truss was found.

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