ISSN 2304-6295

Deformations of the Rod Pyramid: An Analytical Solution

Kirsanov, Mikhail Nikolaevich[1]

[1] Moscow Power Engineering Institute, Moscow, Russian Federation
Correspondence: email c216@ya.ru; contact phone +79651833534

Keywords:

Truss; Pyramid; Analytical solution; Deflection; Maple, Induction

Abstract:

The object of the study is a spatial statically determinate pyramid-type covering truss. The truss has vertical support posts along the perimeter of the base. The corner nodes are fixed on spherical support, cylindrical, and rack. The truss has axial symmetry. The aim is to determine the analytical dependence of the deflection of the structure on the number of panels in its base. Two types of loads are considered: distributed along the edges and vertical loads concentrated at the vertex. ***Method*.** The Maxwell - Mohr formula is used to determine the deflection. The forces in the rods, together with the reactions of the supports, are found in their general system of the equilibrium equation of all nodes. The generalization of partial solutions to an arbitrary number of panels is obtained by induction using operators of the Maple computer mathematics system. ***Results*.** The dependence of the deflection on the number of panels is obtained in the form of a compact formula containing quadratic or linear polynomials in the number of panels. The inclined and horizontal asymptotes of the solutions are found. The existence of deflection minima depending on the number of panels is shown.

1 Introduction

To calculate the stress-strain state of building structures, engineers traditionally resort to numerical methods [1]–[3]. Numerical methods, especially those that use finite element theory, allow us to take into account complex loads, calculate statically indeterminate structures, material inhomogeneities, and other features of the problem. At the same time, numerical methods are limited for calculating structures with a large number of discrete elements, for example, spatial trusses. The limitation is primarily related to the accuracy of the calculations. The well-known "curse of dimensionality", associated with the inevitable accumulation of rounding errors both at the modeling stage (entering design data) and at the calculation stage, does not give full confidence in the accuracy, and sometimes the correctness of the results obtained. One possible approach, in addition to simply increasing the power of computing resources, is to replace the discrete object model with a continuous model. So often, the problem of the deflection or vibration frequency of the beam truss is replaced by the problem of the beam [4]. This, of course, reduces the value of the solution, especially if the practice is interested in the work of a particular structural element, for example, a rod, and not the integral characteristic of the structure (deflection, strength, vibration frequency, etc.). With the development of computer mathematics systems (Maple, Maxima, Mathematica, etc.), the engineer has the opportunity to obtain analytical solutions. The simplest way to replace the numerical input data entered into the mathematical model of the construction with parametric data often allows you to get the desired formula. The value of such a formula is determined by the number of independent design parameters taken into account in it. The most difficult is to obtain formulas for regular systems with a periodic structure when taking into account the number of elements of periodicity. For regular trusses, the periodicity element can be the number of panels. General questions of the problem of the existence and calculation of regular statically determinate constructions are analyzed in [5], [6]. The issues of calculating the strength, stability, and vibrations of regular multi-panel rod frames, trusses, and composite rods are discussed in [7], [8]. Solutions to the problems of deflection of flat trusses as a function of the number of panels by induction using the operators of the Maple system are obtained in [9]–[16]. Analytical calculations of deflections and natural frequencies of spatial trusses are known [17]–[19]. In this paper, we propose a scheme and an analytical calculation of the deflection of a pyramid-type spatial truss structure (Fig. 1). The simplest pyramid trusses are used in models of structures of layered materials [1]–[3], [20].

2 Materials and Methods

The structure of the truss consists of three identical planar triangular lattices with supports on the sides of the base. The bars of the lattices do not connect at the crossing points, the hinges are located only along the edges of the coating. The bases of the faces consist of n horizontal rods of length a. The height of the pyramid is nh. The total number of rods in the structure, including 3 n vertical supports and three rods simulating spherical support joint at corner *A* and a cylindrical one at corner *B*, is equal to $m=18n-6$. The total number of internal hinges in the structure is three times less and equal $q=6n-2,\ m=18n-6$. In this case, the number of hinge nodes on the side edges of the pyramid is $3n-2$, the number of hinges on the sides of the base is 3n. The truss is statically determinate. The task is to find the dependence of the deflection on the number n. To calculate the deflection of a structure whose rods operate in an elastic region, Maxwell – Mohr's formula containing the forces in the rods is used. We will calculate the forces in an analytical form in the program [21], compiled in the language of computer mathematics Maple [22], [23]. The coordinates of the nodes are entered into the program. The origin of the coordinates is selected in support *A*. The coordinates are entered for an arbitrary number n.

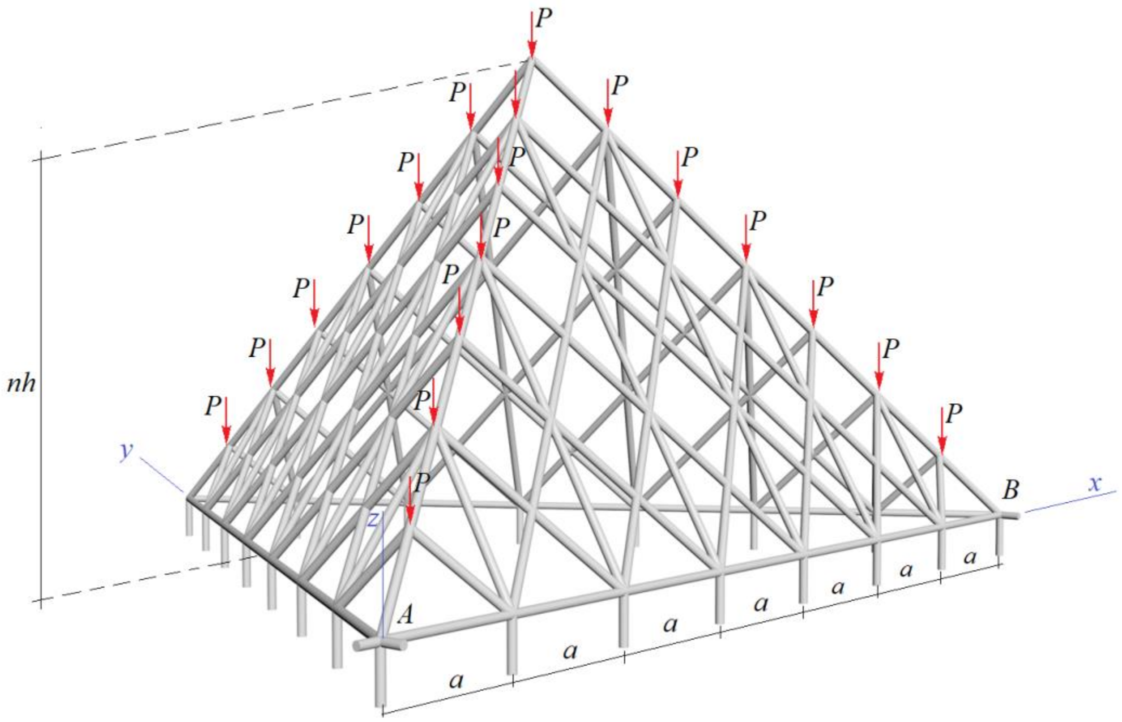

Fig. 1. Truss, *n=7*

The coordinates of the base nodes, for example, have the form:

$$x_i=a(i-1),\ y_i=0,$$

$$x_{i+n}=a(n-(i-1)/2),\ y_{i+n}=a(i-1)\sqrt{3}/2,$$

$$x_{i+2n}=a(n-i+1)/2,\ y_{i+2n}=a(n-i+1)\sqrt{3}/2,\ i=1,...,n,$$

$$z_i=0,\ i=1,...,3n.$$

The order of connecting the rods at the nodes is set by special lists of vertex numbers at the ends of the rods. The forces are obtained by the method of cutting out the nodes. In a cycle, the matrix $\mathbf{G}$ of the system of equilibrium equations of nodes in the projection on the coordinate axis is compiled by the number of nodes

$$\mathbf{GS} = \mathbf{T}.$$

Here $\mathbf{S}$ is the vector of all forces in the rods, including the reactions of the supports. Each node is assigned three rows of the matrix. In the right part of the system $\mathbf{T}$ – the vector of loads on the nodes.

3 Results and Discussion

3.1 The formula for the deflection

We consider a load of intensity *P*, evenly distributed over the nodes of the side edges of the pyramid (Fig. 1). In this case, the non-zero elements of the load vector have the form

$$T_{3i} = P,\ i = 3n+1,...,6n-2. \quad (1)$$

The rows T_{3i-2}, $i = 1,...,q$ contain loads on node i in the projection on the x-axis, and the rows T_{3i-1}, $i = 1,...,q$ contain loads in the projection on the y-axis. In Figure 2, in the projection on the x-y plane, the stretched rods are highlighted in red, and the compressed ones are highlighted in blue. The calculation was carried out at a = 3 m, h = 2m. The thickness of the segments is proportional to the force modules, the number shows the force value related to the value of P and rounded to tenths or hundredths. The contour of the lower belt was stretched, the side ribs were compressed. For the most stretched rods, the method of induction can be used to obtain analytical expressions of the force depending on the number of panels. Next, consider the case of an odd number of panels in the base n=2k+1.

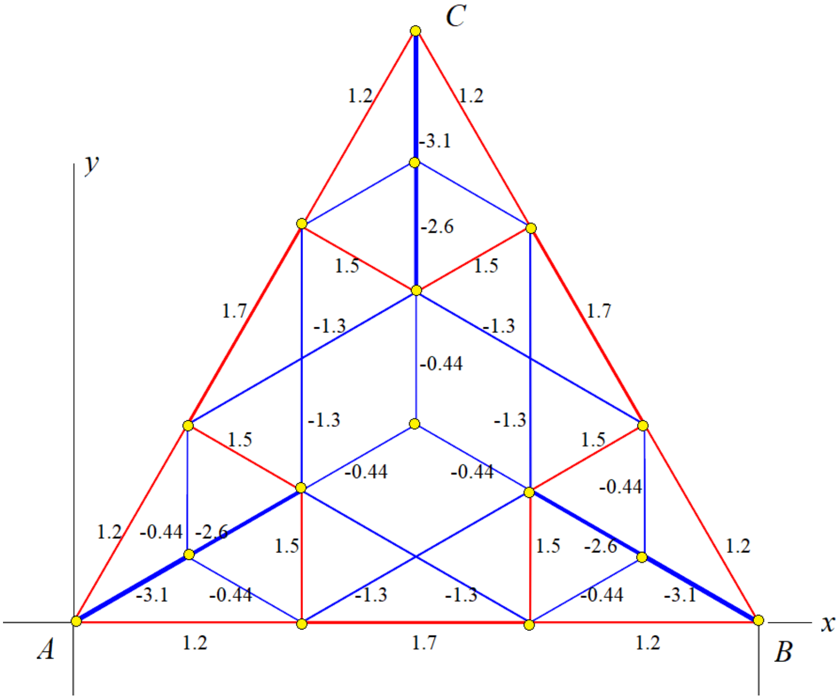

Fig. 2. Distribution of forces on the rods of the pyramid, *n* =3

Depending on the parity of *k*, the middle rod II or the adjacent rod I turn out to be the most stretched in the lower contour alternately (Fig. 3). From the calculation of only six trusses with a consistently increasing number of *k*, we get the general terms of the sequences of forces:

$$S_I = Pa(5+3(-1)^k+12k)/(18h), \quad (2)$$

$$S_{II} = Pa(5-3(-1)^k+12k)/(18h).$$

Similarly, we obtain the force for the most compressed rods at the bottom of the side edges:

$$S_{III} = -Pc(6k+1)/(9h),\ c = \sqrt{3a^2+9h^2}. \quad (3)$$

The resulting formulas show that, as expected, as the height h decreases, the forces increase. Somewhat unexpectedly, it turned out that the reactions of all vertical supports on the sides of the base, except for the corners *A*, *B*, and *C*, are equal to zero (Fig. 3). The reactions of the angular supports with uniform loading of the internal hinges by vertical forces P are equal to $Z_A = Z_B = Z_C = P(3n-2)/3$.

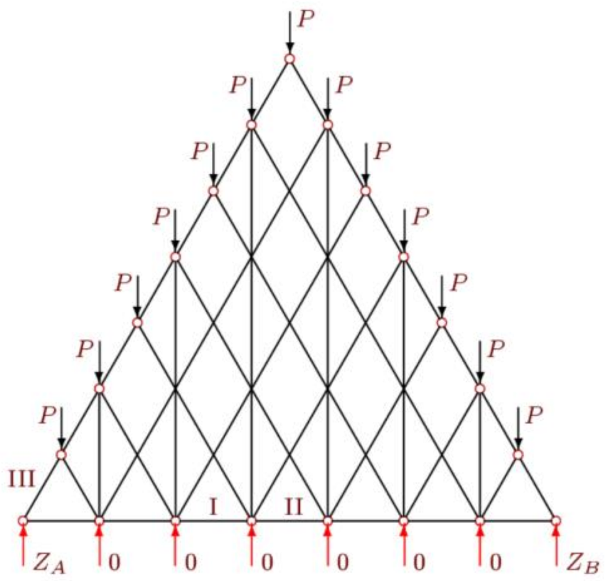

Fig. 3. The side face of the pyramid and the reaction of the supports, *n*=7

The deflection of the vertex under the action of the load is calculated by Maxwell-Mohr's formula

$$\Delta_k = \sum_{\alpha=1}^{m-3n-3} S_\alpha^{(P)} S_\alpha^{(1)} l_\alpha / (EF), \tag{4}$$

Here it is indicated $S_\alpha^{(1)}$ — the forces in the element number α from the action of a single vertical force applied to the top of the pyramid, $S_\alpha^{(P)}$ — the force from the action of a distributed load of intensity P, EF — the longitudinal stiffness of the rods, l_α — the lengths of the rods. Summation is carried out for all truss rods, except for 3*n*–3 reference ones, which are accepted as non-deformable. Calculations of trusses with different numbers of panels in the base show that the result has the same shape, regardless of k

$$\Delta_k = (C_1 a^3 + C_2 c^3) / (h^2 EF), \tag{5}$$

where the coefficients C_1 and C_2 depend on *k*. These dependencies are determined by induction. We have the following sequence of solutions

$$\Delta_1 = 2(36a^3 + 7c^3) / (81h^2 EF),$$
$$\Delta_2 = (71a^3 + 13c^3) / (27h^2 EF),$$
$$\Delta_3 = 2(213a^3 + 38c^3) / (81h^2 EF),$$
$$\Delta_4 = (711a^3 + 125c^3) / (81h^2 EF),$$
$$\Delta_5 = 2(178a^3 + 31c^3) / (27h^2 EF),$$

...

The common terms of the coefficient sequences can be found using the operators of the Maple or online system [24]. We get the following solution:

$$C_1 = (12k^2 + 11k + 1) / 27, \; C_2 = (k+1)(6k+1) / 81. \tag{6}$$

Thus, solution (5) with the coefficients (6) represents the desired dependence of the deflection on the load, the size of the pyramid, and the number of panels. Similarly, the formula for the dependence of the deflection on the action of the vertical force only on the top of the pyramid is derived. As in the previous case, the reactions differ from zero only in the corner supports, and the coefficients C_1 and C_2 have the form

$$C_1 = (2k+1) / 27, \; C_2 = (2k+1) / 81. \tag{7}$$

3.2 Numerical example

Consider a pyramid with a base side length $L = na$ under the action of a uniform load. We introduce a dimensionless relative deflection $\Delta' = EF\Delta / (P_s L)$, where $P_s = P(3n-2)$ is the total vertical load on the structure. At *L* = 80 m, the dependence of the relative deflection on the number of panels is represented by the curves in Figure 4. In all cases, the resulting relationship reveals a minimum that falls on the number of panels, which increases with decreasing height. For high pyramids, the extremum is more pronounced. The asymptotes of the constructed dependence can be detected by calculating the limit: $\lim_{k\to\infty} \Delta'/k = h/(3L)$. In another formulation, if, when changing the number of panels, we fix not only the length of the base but also the height of the pyramid $H = nh$, then the asymptotics will be different $\lim_{k\to\infty} \Delta' = (\sqrt{3}L_1^{\,3} + 2L^3)/(54H^2L)$, where $L_1 = \sqrt{3H^2 + L^2}$. In this case, the minimum on the corresponding curves is not detected, and the relative deflection with an increase in the number of panels almost does not change, differing little from its limit $(\sqrt{3}L_1^{\,3} + 2L^3)/(54H^2L)$.

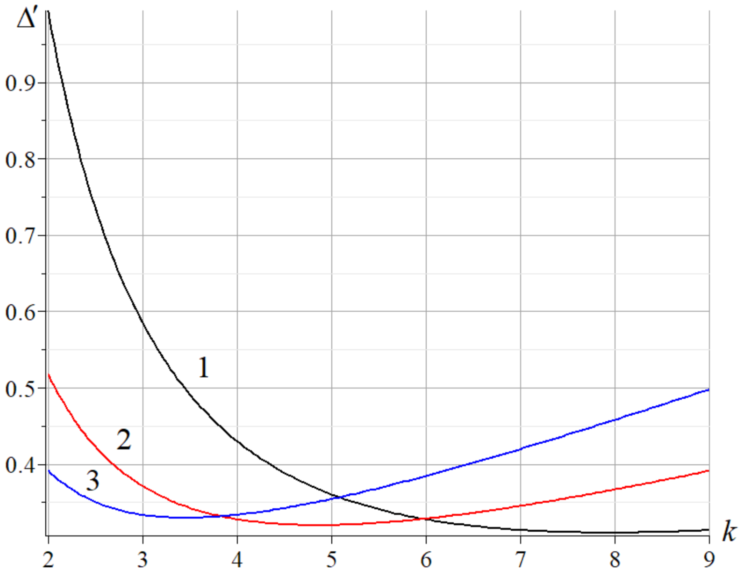

Fig. 4. Dependence of the relative deflection on the number of panels, 1 – *h*=5m; 2– *h*=8m; 3– *h*=11m.

4 Conclusions

The main results of the work are as follows.

1. A statically determinate scheme of a three-dimensional truss of a pyramidal covering is proposed. The forces in the most stretched and compressed elements are determined in the analytical form.

2. It is shown that for a nodal load that is uniform along the edges, the reactions of all supports except the angular ones are zero.

3. The formula for the dependence of the deflection of the top of the pyramid on the number of panels in the base for distributed and concentrated load is derived.

4. Linear asymptotics of solutions are found.

5 Acknowledgements

This research has been supported by the Interdisciplinary Scientific and Educational School of Moscow University «Fundamental and Applied Space Research».

References

1. Santana, M.V.B., Gonçalves, P.B., Silveira, R.A.M. Stability and load capacity of an elasto-plastic pyramidal truss. International Journal of Solids and Structures. 2019. 171. Pp. 158–173.

DOI:10.1016/j.ijsolstr.2019.04.011.
2. Ye, G., Bi, H., Hu, Y. Compression behaviors of 3D printed pyramidal lattice truss composite structures. Composite Structures. 2020. 233. Pp. 111706. DOI:10.1016/j.compstruct.2019.111706.
3. Wang, Y.Z., Ma, L. Sound insulation performance of membrane-type metamaterials combined with pyramidal truss core sandwich structure. Composite Structures. 2021. 260. Pp. 113257. DOI:10.1016/j.compstruct.2020.113257.
4. Liu, M., Cao, D., Zhang, X., Wei, J., Zhu, D. Nonlinear dynamic responses of beamlike truss based on the equivalent nonlinear beam model. International Journal of Mechanical Sciences. 2021. 194. Pp. 106197. DOI:10.1016/j.ijmecsci.2020.106197.
5. Hutchinson, R.G., Fleck, N.A. The structural performance of the periodic truss. Journal of the Mechanics and Physics of Solids. 2006. 54(4). Pp. 756–782. DOI:10.1016/j.jmps.2005.10.008.
6. Hutchinson, R.G., Fleck, N.A. Microarchitectured cellular solids - The hunt for statically determinate periodic trusses. ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik. 2005. 85(9). Pp. 607–617. DOI:10.1002/zamm.200410208.
7. Galishnikova, V.V., Ignatiev, V.A. Regulyarnyye sterzhnevyye sistemy. Teoriya i metody rascheta. [Regular rod systems. Theory and methods of calculation]. Volograd, VolgGASU, 2006. ISBN:5-98276-125-7.
8. Ignatiev, V.A. Raschet regulyarnykh sterzhnevykh sistem [Calculation of regular rod systems]. Saratov, Saratov Higher Military Chemical Military School, 1973. https://www.elibrary.ru/item.asp?id=28958501.
9. Voropay, R. A., Domanov, E. V. The dependence of the deflection of a planar beam truss with a complex lattice on the number of panels in the system Maple. Postulat. 2019. (1). URL: http://e-postulat.ru/index.php/Postulat/article/view/2258/0.
10. Ilyushin, A. The formula for calculating the deflection of a compound externally statically indeterminate frame. Structural mechanics and structures. 2019. 22(3). Pp. 29–38. URL: https://www.elibrary.ru/item.asp?id=41201106.
11. Voropay, R. A., Domanov, E. V. The formula for the dependence of the deflection of a truss with an asymmetric lattice on the number of panels. Postulat. 2018. (6). Pp. 61. URL: http://e-postulat.ru/index.php/Postulat/article/view/1653.
12. Ovsyannikova, V.M. Dependence of deformations of a trapezous truss beam on the number of panels. Structural Mechanics and Structures. 2020. 26(3). Pp. 13–20.
13. Rakhmatulina, A.R., Smirnova, A.A. Two-parameter derivation of the formula. Postulat. 2018. 31(5–1). URL: http://e-postulat.ru/index.php/Postulat/article/view/1456.
14. Rakhmatulina, A.R., Smirnova, A.A. The formula for the deflection of a truss loaded at half-span by a uniform load. Postulat. 2018. 29(3). URL: http://e-postulat.ru/index.php/Postulat/article/view/1293.
15. Kitaev, S.S. Derivation of the formula for the deflection of a cantilevered truss with a rectangular diagonal grid in the computer mathematics system Maple. Postulat. 2018. 5–1. Pp. 43. URL: http://e-postulat.ru/index.php/Postulat/article/view/1477.
16. Voropay, R.A. Derivation of the formula for the deflection of the truss with additional horizontal struts. Postulat. 2018. 6. Pp. 105. URL: http://e-postulat.ru/index.php/Postulat/article/view/2431.
17. Kirsanov, M.N., Vorobyev, O.V. Calculating of a spatial cantilever truss natural vibration frequency with an arbitrary number of panels: analytical solution. Construction of Unique Buildings and Structures. 2021. 94. Pp. 9402. DOI:10.4123/CUBS.94.2.
18. Kirsanov, MN, Tinkov, DV. Analysis of the natural frequencies of oscillations of a planar truss with an arbitrary number of panels. Vestnik MGSU. 2019. 14(4). Pp. 284–292. DOI:10.22227/1997-0935.2019.3.284-292.
19. Kirsanov, M.N., Petrichenko, E.A., Vorobev, O. V. The formula for the lower estimate of the fundamental frequency of natural vibrations of a truss with an arbitrary number of panels. Construction of Unique Buildings and Structures. 2021. 94(1). Pp. 9403–9403. DOI:10.4123/CUBS.94.3.
20. Santana, M.V.B., Gonçalves, P.B., Silveira, R.A.M. Closed-form solutions for the symmetric nonlinear free oscillations of pyramidal trusses. Physica D: Nonlinear Phenomena. 2021. 417. Pp. 132814. DOI:10.1016/j.physd.2020.132814.
21. Buka-Vaivade, K., Kirsanov, M.N., Serdjuks, D.O. Calculation of deformations of a cantilever-frame planar truss model with an arbitrary number of panels. Vestnik MGSU. 2020. 4(4). Pp.

510–517. DOI:10.22227/1997-0935.2020.4.510-517.
22. Zotos, K. Performance comparison of Maple and Mathematica. Applied Mathematics and Computation. 2007. 188(2). Pp. 1426–1429. DOI:10.1016/j.amc.2006.11.008.
23. Rapp, B.E. Introduction to Maple. Microfluidics: Modelling, Mechanics and Mathematics. Elsevier, 2017. Pp. 9–20.
24. Wolfram|Alpha: Computational Intelligence. URL: https://www.wolframalpha.com/ (date of application: 4.07.2021).