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# Model of a spatial dome cover. Deformations and oscillation frequency

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**Abstract:**

**The object of research.** A new scheme of a statically determinate spatial truss is considered. The design has a hexagonal dome resting on two belts. The belts are supported by vertical racks. Two corner supports have spherical and cylindrical hinges. The outer support contra consists of 6n horizontal rods, the inner one consists of *6(n-1)* rods. The contours are connected by skews. Formulas are derived for the deflection of the vertex and the angular hinge depending on n. The upper and lower analytical estimates of the first frequency of natural oscillations of the structure are found. **Method**. Calculation of the forces in the rods is carried out by cutting out the nodes from the solution of the system of equilibrium equations for all nodes in the projection on the coordinate axes. To derive formulas for the dependence of breakdowns, forces, and the frequency of free oscillations, an inductive generalization of the sequence of solutions for structures with a different number of panels is used. The structural stiffness matrix and deflection are calculated using the Maxwell - Mohr formula in analytical form. To find estimates of the lowest frequency of vibrations of nodes endowed with masses, the Dunkerley and Rayleigh methods are used. **Results**. The vertical load distributed over the nodes and the concentrated load applied to the top are considered. Formulas for the forces in the characteristic bars of the structure are derived. A picture of the distribution of forces throughout the structure is presented. The resulting formulas for the deflection and frequency estimates have a compact form. The upper estimate of the first oscillation frequency of nodes under the assumption of vertical displacements of points has fairly high accuracy. The analytical solution is compared with the lowest oscillation frequency obtained numerically. All analytical transformations are performed in the Maple symbolic mathematics system. Some asymptotics of solutions is found.

## 1 Introduction

Spatial truss structures are traditionally used in large-span roof structures of public buildings, trade, and transport enterprises. It is not always possible to represent a spatial structure as a certain sum of planar trusses, for which both numerical [1]–[6] and analytical methods of calculation are well developed. One of these structures is discussed in this article. Among planar and spatial trusses, a class of regular trusses can be distinguished separately. For such trusses, inductive calculation methods are possible to obtain a solution that is valid for an arbitrary number of structure periodicity elements.

Hutchinson R.G. and Fleck N.A. [7], [8] were among the first to deal with the problem of the existence and calculation of statically determinate regular systems. In the monographs A. Kaveh [9], [10], methods of graph theory and group theory are used to solve optimization problems for regular and symmetric large-scale constructions. It is shown that these methods can significantly reduce the calculations of such systems.

Analytical methods and algorithms using the superposition method and representation in the form of trigonometric series, implemented in the Maple system, are proposed in [11]–[13].

The handbooks [14], [15] contain analytical solutions for the deflection of various schemes of planar statically determinate regular trusses under various loads. To derive the formulas, the operators of the Maple symbolic mathematics system and the induction method were used, which consists in generalizing a series of separate calculations of trusses with a successively increasing number of panels to the case of an arbitrary number of panels.

Separate analytical solutions obtained by induction are also known for deflection problems for planar trusses [16], arches [17], [18], and frames [19]–[21]. A similar algorithm was used in calculations of deformations of three-dimensional trusses [22] and in deriving a formula for estimating the first frequency of natural oscillations of statically determinate trusses [23]–[25].

In this paper, a new scheme for a spatial regular statically determinate structure of a dome cover was proposed. The task was set to derive analytical dependences of truss deformations, forces in critical rods, and oscillation frequency on the number of panels. The formulas obtained can be used to evaluate numerical solutions, especially for large-scale structures, in the numerical calculations in which the accumulation of rounding errors, leading to loss of calculation accuracy, is inevitable.

# 2 Materials and methods

## 2.1 Design of the truss

The coating structure consists of two closed rod contours. A hexagonal rod pyramid rests on the outer contour (Fig. 1). Contours connect braces of length $c=\sqrt{a^2+h^2}$ . The outer contour contains $6n$ rods of length *a*, the inner one contains $6(n-1)$ rods. The length of the edge of the pyramidal dome is equal to $nc$. The outer contour of the structure is fixed on vertical supports with a height $h$. There are three additional horizontal bonds at corners *A* and *B*, so that corner *A* is a spherical hinge and corner *B* is cylindrical. The whole structure consists of $n_s=36n-15$ rods, including posts and rods modeling supports.

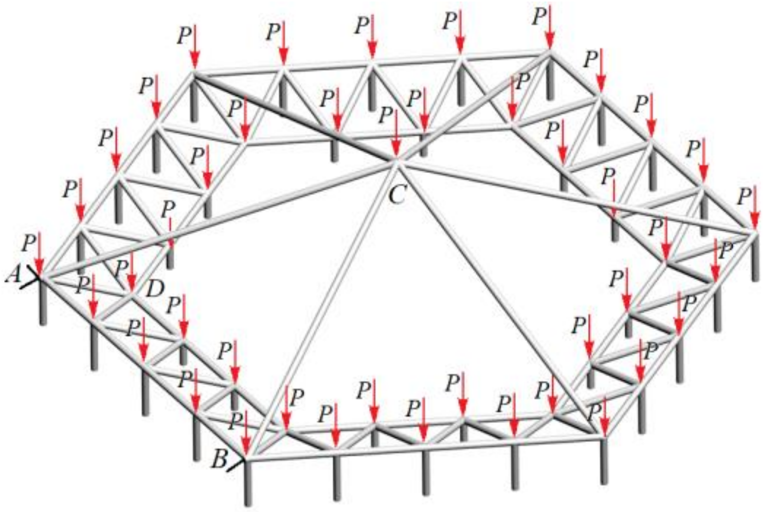


**Fig. 1- Coating structure under uniform load *n*=4**

## 2.2 Force Calculation Algorithm

The design is statically determined, the number of internal truss nodes is three times less than the number of rods: $K=12n-5$. The truss is loaded with vertical forces uniformly distributed over the nodes. The forces in a program [26] written in the Maple language will be calculated. The program includes a method for cutting nodes. To enter information about the structure of the structure into the program, the nodes are numbered (Fig. 2) and their coordinates are entered. The coordinates of the outer contour have the form:

$$x_{i+jn}=L\cos\theta-a(i-1)\cos\beta,$$
$$y_{i+jn}=L\sin\theta+a(i-1)\sin\beta,$$
$$z_{i+jn}=h,\ i=1,..,n,\ j=0,..,5,$$

where $L = na,\ \theta = j\pi/3,\ \beta = \pi/3 - \theta.$

The coordinates of the hinges of the inner (lower) contour:

$$x_{i+j(n-1)+6n} = (L-a)\cos\theta - a(i-1)\cos\beta,$$
$$y_{i+j(n-1)+6n} = (L-a)\sin\theta + a(i-1)\sin\beta,$$
$$z_{i+j(n-1)+6n} = 0,\ i = 1,..,n-1,\ j = 0,..,5.$$

The top of the dome *C* has the following coordinates:

$$x_{12n-5} = y_{12n-5} = 0,\ z_{12n-5} = (n+1)h.$$

The coordinates of the hinges to which the support posts of the outer contour are attached from below:

$$x_{i+12n-5} = x_i,\ \ y_{i+12n-5} = y_i,\ z_{i+12n-5} = -h,\ i = 1,..,6n.$$

The coordinates of the lower hinges of the racks of the internal contour:

$$x_t = x_{i+(5+j)n-j+2}, y_t = x_{i+(5+j)n-j+2},\ \ z_t = -h,$$
$$t = i + 18n + (j-1)(n-2) - 5,\ \ i = 1,..,n-2,\ j = 1,..,6.$$

To determine the structure of the lattice of rods, oriented lists of vertices of the ends of the rods are introduced $\Phi_i,\ i = 1,..,n_s$. Bars of the outer contour, for example, are specified by lists: $\Phi_i = [i, i+1],\quad i = 1,..,6n-1,\ \ \Phi_{6n} = [6n, 1]$. Lists of end numbers of bars of the inner contour look like this:

$$\Phi_{i+6n} = [i+6n, i+6n+1],\ i = 1,..,6n-7, \Phi_{12n-6} = [12n-6, 6n+1].$$

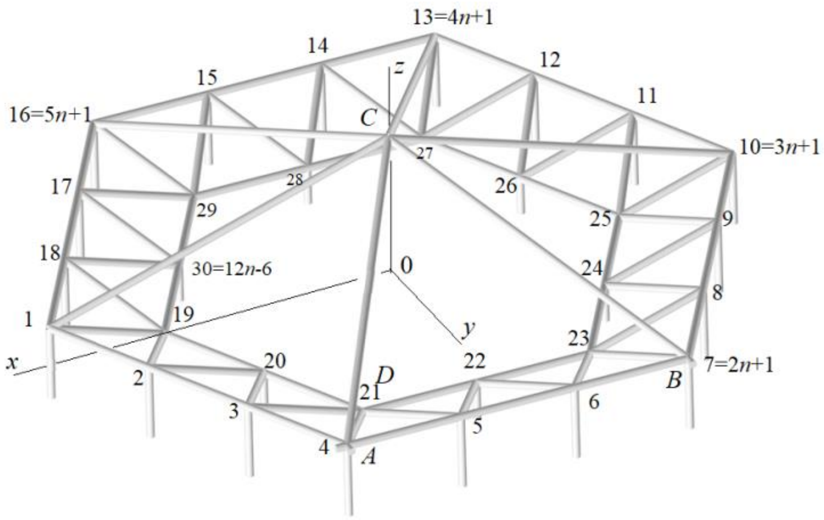


**Fig. 2 - Truss node numbering, *n*=3**

The matrix $\mathbf{G}$ of coefficients of the system of linear equations of equilibrium of nodes in the projection on the coordinate axes are calculated from the coordinate values and lists of the ends of the rods:

$$g_{v,i} = (v_{\Phi_{i,1}} - v_{\Phi_{i,2}})/l_i,\ \ v = x, y, z,\ \ i = 1,...,n_s + 3,$$

where $l_i = \sqrt{\sum_{v=x,y,z} (v_{\Phi_{i,1}} - v_{\Phi_{i,2}})^2}$ — the length of the rod with number *i*. The number of bars also includes the supporting corner bars at vertices *A* and *B*. The coefficient matrix is filled in by rows. Every three lines correspond to the direction cosines of the forces with the *x, y, z* axes, respectively:

$$G_{3\Phi_{i,1}-2+j,i} = g_{j,i}, G_{3\Phi_{i,2}-2+j,i} = -g_{j,i},\ \ j = 1,2,3,\ i = 1,..,n_s.$$

It takes into account the fact that the force vector applied to the node at one end of the rod is opposite to the force vector applied to the node at the other end of the rod.

The system of equilibrium equations for nodes has a matrix form $\mathbf{GS} = \mathbf{T}$, where $\mathbf{S}$ is the column vector of length $n_s$ consists of the values of unknown forces in the rods, including the reactions of the supports. The load vector $\mathbf{T}$ has the same length. The projections on the x-axis of the loads applied to node *i* are written in the elements of this vector $T_{3i-2}$, on the *y*-axis – in the elements $T_{3i-1}$. The values of vertical external forces are contained in the elements $T_{3i}$, $i=1,...,K$. The solution of the matrix equation in symbolic form in the Maple symbolic mathematics system is performed by the inverse matrix method $\mathbf{S}=\mathbf{G}^{-1}\mathbf{T}$.

# 3 Results

## 3.1 Forces in rods

The picture of the force distribution over the structure bars for the case of a uniform vertical external load $a=6\text{m}$, $h=1\text{m}$ is shown in Figure 3. The force values are related to the load on node *P* and rounded into two significant figures.

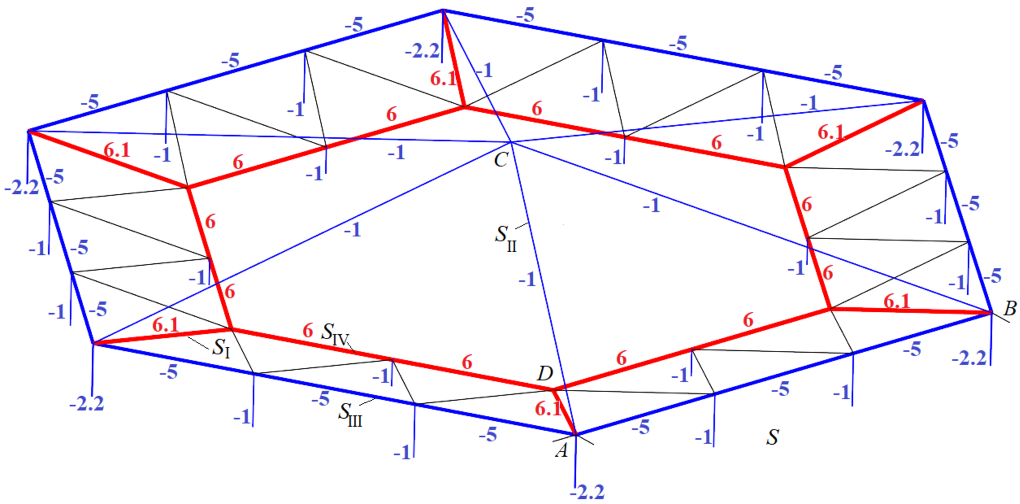

**Fig. 3 - Distribution of forces in the truss rods, *n*=3**

Compressed rods are highlighted in blue, stretched rods are highlighted in red. The thickness of the segments of the rods is conditionally proportional to the force modules. The braces marked with thin black lines are not stressed. The inner contour under such a load is stretched, the outer one is compressed.

Based on the results of force calculation, formulas for forces in characteristic bars can be derived. A rare feature for regular systems was noticed: the forces in the corresponding rods do not depend on the number of panels. For any order of the truss, the forces in the corner braces are always the same: $S_{\text{I}}=Pc/h$. Compressive forces in the six rods of the dome: $S_{\text{II}}=Pc/(6h)$, the forces of the rods of the outer contour $S_{\text{III}}=-5Pa/(6h)$, the forces in the inner contour $S_{\text{IV}}=Pa/h$. The forces in all support posts on the outer contour, except for the corner ones, are equal $-P$. The forces of the racks on the inner contour are also equal $-P$. The vertical reactions of the corner supports do not depend on the dimensions of the truss and are calculated by the formula $R_{coner}=13P/6$. The resulting solution is easily verified. The sum of projections on the z-axis of all reactions of supports of external forces applied to the $K=12n-5$ nodes of the structure is equal to zero:

$$6(n-1)\cdot P+6(n-2)\cdot P+6R_{coner}-(12n-5)\cdot P=0.$$

## 3.2 Deflection

When calculating the deflection at individual points of the truss and in the future, when calculating the natural oscillation frequency, we will use the Maxwell-Mohr formula, assuming that all the rods are linearly elastic and work only in compression and tension:

$$\Delta=\sum_{\alpha=1}^{n_s}S_\alpha^{(P)}S_\alpha^{(1)}l_\alpha/(EF) \tag{1}$$

The sum is made for all deformable bars of the structure, including support bars. Standard designations are used here: *E* is the modulus of elasticity of the material, *F* is the cross-sectional area of the rod, $S_\alpha^{(P)}$ is the force from the action of an external load in the rod with the number $\alpha$, $S_\alpha^{(1)}$ is the force in the same rod from the action of a unit vertical force applied to the vertex *C*, $l_\alpha$ is the long rod.

Let's find the deflection for several trusses with a successively increasing number of panels.

$$\Delta_2 = -P(5a^3 - c^3 - 13h^3)/(3h^2EF),$$
$$\Delta_3 = -P(15a^3 - 3c^3 - 26h^3)/(6h^2EF),$$
$$\Delta_4 = -P(10a^3 - 2c^3 - 13h^3)/(3h^2EF),$$
$$\Delta_5 = -P(25a^3 - 5c^3 - 26h^3)/(6h^2EF),$$
$$\Delta_6 = -P(15a^3 - 3c^3 - 13h^3)/(3h^2EF),...$$

Generalizing these formulas by methods of the Maple system to an arbitrary number $n$, it is obtained:

$$\Delta_n(C) = -P(5na^3 - nc^3 - 26h^3)/(6h^2EF). \quad (2)$$

Usually, in similar problems, [16-21] the procedure for obtaining a general solution for several private ones requires the involvement of special operators of computer mathematics. In this case, the dependencies are very simple and linear in the number of panels.

Similarly, we obtain a dependence of the same form for the deflection of the angular (not supported) node *D*:

$$\Delta_n(D) = -P((11n-6)a^3 + 6c^3 + 26h^3)/(6h^2EF). \quad (3)$$

The form of the solution turned out to be quite simple. Compared to polynomial dependences on the number of panels of degrees four or more in similar solutions for planar systems [16-21], linear solutions (2) and (3) are much simpler.

Let us introduce the value of the dimensionless deflection $\Delta' = EF\Delta_k(n)/(P_0L)$, referred to as the length of the side edge of the outer contour of the truss $L = na$ and the total load $P_0 = (12n-5)P$. The found dependences of the deflections at various lateral points on the console have a limiting value on the horizontal asymptote: $\lim_{n\to\infty}\Delta'(C) = 0$, $\lim_{n\to\infty}\Delta'(D) = h/(72L)$ (Fig. 4).

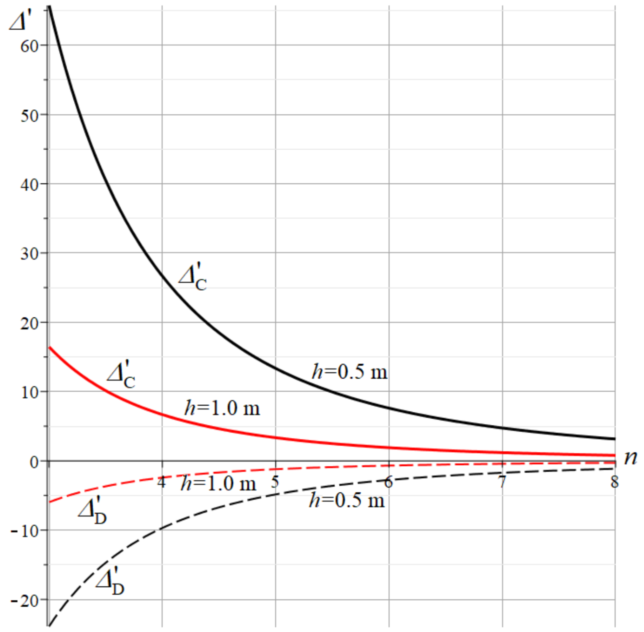


**Fig. 4 - Relatie deflections at points *C* and *D* at** $L = an = 40\text{m}$

Kirsanov, M.;
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The curves of the dependences of deflections (2) and (3) on the height $h$ at a certain value of the height intersect (Fig. 5). Equating (2) and (3) we obtain an equation whose solution can be found analytically:

$$h^* = \frac{L}{n}\sqrt{\sqrt[3]{4(8n-3)^2/(n-6)^2}-1}\,.$$

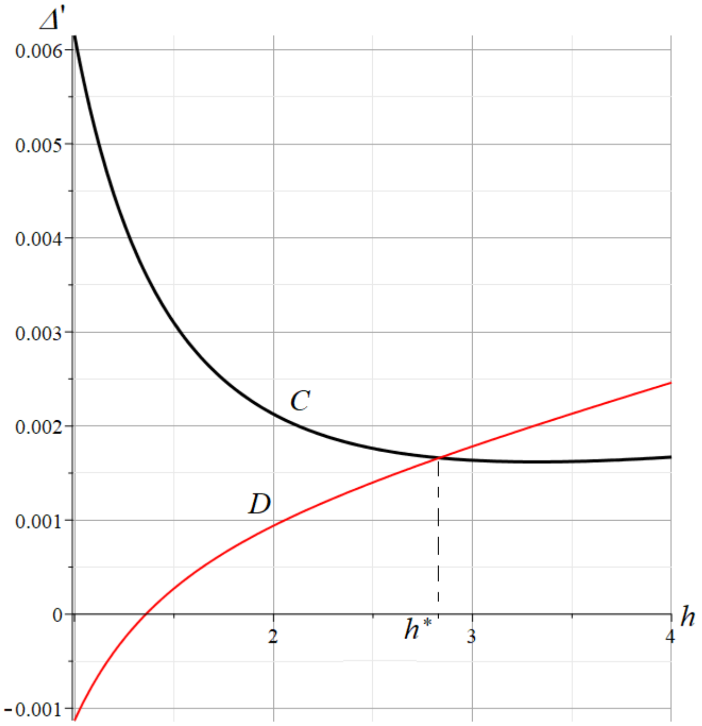


**Fig. 5 - Relatie deflections at points *C* and *D* at** $L = an = 40\text{m},\ n = 35$

These curves also have asymptotes, the angle tangents of which can be found using Maple:

$$\lim_{h\to\infty}\frac{\Delta'_C}{h} = \frac{16}{L(36n-15)},\quad \lim_{h\to\infty}\frac{\Delta'_D}{h} = \frac{26+n}{L(72n-30)}.$$

## 3.3 Natural frequency

Most dynamic calculations of structures include the calculation of the value of the first (lowest) frequency of natural oscillations. As a rule, the calculation of natural frequencies is performed numerically based on various variants of the finite element method [27]–[29]. Analytical methods are only possible for the upper and lower frequency estimates [24]. Approximate methods for obtaining such estimates are known [30], [31]. These methods are based on the calculation of partial frequencies, the values of which can in some cases be found analytically. For regular constructions, analytical estimates can be generalized to an arbitrary number of panels using the [25] induction method.

The inertial properties of the structure under consideration by the same concentrated masses m in the nodes are modeled. Neglecting the horizontal displacements of the weights, only their vertical vibrations will be considered. The number of degrees of freedom of the truss weight system of order $n$ is equal to the number of nodes $K = 12n - 5$.

The differential equations for the movement of goods in matrix form were written:

$$\mathbf{M}_K\ddot{\mathbf{Z}} + \mathbf{D}_K\mathbf{Z} = 0, \tag{4}$$

where $\mathbf{Z}$ – the vector of vertical displacements of masses 1,..., *K*, $\mathbf{M}_K$ is the inertia matrix of size $K \times K$, $\mathbf{D}_K$ is the stiffness matrix, $\ddot{\mathbf{Z}}$ is the acceleration vector. In the case of equal masses, the inertia matrix is proportional to the identity matrix $\mathbf{M}_K = m\mathbf{I}_K$. The elements of the matrix $\mathbf{B}_K$ inverse to the stiffness matrix $\mathbf{D}_K$ can be found using the Maxwell-Mohr formula:

$$b_{i,j}=\sum_{\alpha=1}^{n_s}S_\alpha^{(i)}S_\alpha^{(j)}l_\alpha/(EF), \quad (5)$$

where, by analogy with (1), the designation is introduced here: $S_\alpha^{(i)}$ is the force in the rod $\alpha$ from the action of a unit vertical force at node *i*. The problem is reduced to the problem of matrix eigenvalues $\mathbf{B}_K$. We multiply (4) by $\mathbf{B}_K$ on the left, and taking into account $\ddot{\mathbf{Z}}=-\omega^2\mathbf{Z}$ the replacement corresponding to harmonic oscillations:

$$z_i=u_i\sin(\omega t+\varphi_0) \quad (6)$$

Thus, we obtain the equation of the problem for the eigenvalues of the matrix $\mathbf{B}_K\mathbf{Z}=\lambda\mathbf{Z},$ where $\lambda=1/(m\omega^2)$ is the eigenvalue of the matrix $\mathbf{B}_K$, $\omega$ is the natural oscillation frequency. From here, the frequency of natural oscillations has the form $\omega=\sqrt{1/(m\lambda)}$. The eigenvalue problem is solved by the Eigenvalues operator of the Maple system.

The elements of the matrix $\mathbf{B}_K$ depend on the forces $S_\alpha^{(i)}$ in the rods, which are found from the solution of the system of equations of the truss nodes in projections onto three coordinate axes. The same system also includes the reactions of the supports.

Consider two approximate methods that give upper and lower bounds for the first frequency.

### 3.3.1 The energy method. Top rating

From the equality of the maximum values of the kinetic and potential energies:

$$T_{\max}=\Pi_{\max} \quad (7)$$

follows the Rayleigh formula for the upper estimate of the first frequency. The kinetic energy of the system of all masses m located at the nodes of the structure has the form:

$$T=\sum_{i=1}^{K}mv_i^2/2.$$

The vertical velocity of the mass i according to (6) has the form:

$$v_i=\dot{z}_i=\omega u_i\sin(\omega t+\varphi_0).$$

Assuming that at maximum kinetic energy $\max(\sin(\omega t+\varphi_0))=1$, we get:

$$T_{\max}=\omega^2 m\sum_{i=1}^{K}u_i^2/2, \quad (8)$$

where the amplitude of the vertical displacement is calculated using the Maxwell-Mohr formula:

$$u_i=\sum_{\alpha=1}^{n_s}S_\alpha^{(P)}\tilde{S}_\alpha^{(i)}l_\alpha/(EF)=P\sum_{\alpha=1}^{n_s}\tilde{S}_\alpha^{(P)}\tilde{S}_\alpha^{(i)}l_\alpha/(EF)=P\tilde{u}_i.$$

The previous designations are used: $S_\alpha^{(P)}$ - force in the rod $\alpha=1,...,n_s$ from the action of the load *P*, uniformly distributed over the nodes, $\tilde{S}_\alpha^{(i)}$ — force in the same rod from a single (dimensionless) load applied to the mass with number *i*, $\tilde{S}_\alpha^{(P)}=S_\alpha^{(P)}/P$. The form of vibrations of the system of loads with the first frequency is close to the form of deflection of the structure from a uniform load. Thus, (8) takes the form:

$$T_{\max}=P^2\omega^2\sum_{i=1}^{K}m\tilde{u}_i^2/2, \quad (9)$$

where $\tilde{u}_i = u_i / P = \sum_{\alpha=1}^{n_s} \tilde{S}_\alpha^{(P)} \tilde{S}_\alpha^{(i)} l_\alpha / (EF)$ is the amplitude of displacements of the mass with a number *i* under the action of a distributed load, referred to as the value of *P*.

Let us write the potential energy of deformation of elastic rods:

$$\Pi_{\max} = \sum_{\alpha=1}^{n_s} S_\alpha^{(P)} \Delta l_\alpha / 2 = \sum_{\alpha=1}^{n_s} (S_\alpha^{(P)})^2 l_\alpha / (2EF).$$

Due to the linearity of the load problem, we have $S_\alpha^{(P)} = P\sum_{i=1}^{N} \tilde{S}_\alpha^{(i)}$.

$$\Pi_{\max} = P^2 \sum_{\alpha=1}^{n_s} \tilde{S}_\alpha^{(P)} \sum_{i=1}^{K} \tilde{S}_\alpha^{(i)} l_\alpha / (2EF) = P^2 \sum_{i=1}^{K} \sum_{\alpha}^{n_s} \tilde{S}_\alpha^{(P)} \tilde{S}_\alpha^{(i)} l_\alpha / (2EF) = P^2 \sum_{i=1}^{N} \tilde{u}_i / 2. \quad (10)$$

From (7), (9), (10) follow the Rayleigh formula for the upper estimate of the first oscillation frequency of the truss:

$$\omega_R^2 = \sum_{i=1}^{K} \tilde{u}_i / \sum_{i=1}^{K} m\tilde{u}_i^2. \quad (11)$$

Generalizing a series of solutions for displacement $\tilde{u}_i$ at various *n*, we find the dependence of the frequency on the construction order *n*. Consider separately the sums $\sum_{i=1}^{K} \tilde{u}_i$ and $\sum_{i=1}^{K} \tilde{u}_i^2$.

The calculation of displacement for trusses with different numbers of panels shows that the solution for the sums $\sum_{i=1}^{K} \tilde{u}_i$ in the numerator (11) has the form:

$$\sum_{i=1}^{K} \tilde{u}_i = (C_a a^3 + C_c c^3 + C_h h^3) / (h^2 EF),$$

or in a more compact form:

$$\sum_{i=1}^{K} \tilde{u}_i = \sum_{\alpha=[a,c,h]} m g_\alpha \alpha^3 / (h^2 EF), \quad (12)$$

where the coefficients $C_a, C_c, C_h$ are obtained by induction, generalizing a series of solutions for different *n:*

$$n = 2,\ \sum_{i=1}^{K} \tilde{u}_i = (134a^3 + 50c^3 + 205h^3) / (6h^2 EF),$$

$$n = 3,\ \sum_{i=1}^{K} \tilde{u}_i = (219a^3 + 51c^3 + 313h^3) / (6h^2 EF),$$

$$n = 4,\ \sum_{i=1}^{K} \tilde{u}_i = (304a^3 + 52c^3 + 421h^3) / (6h^2 EF),$$

$$n = 5,\ \sum_{i=1}^{K} \tilde{u}_i = (389a^3 + 53c^3 + 529h^3) / (6h^2 EF),$$

$$n = 6,\ \sum_{i=1}^{K} \tilde{u}_i = (474a^3 + 54c^3 + 637h^3) / (6h^2 EF), \ldots$$

As a result, we have coefficients:

$$C_a = (85n - 36),\ C_c = (n + 48)/6,\ C_h = (108n - 11)/6. \quad (13)$$

The denominator (12) has a more complex form:

$$\sum_{k=1}^{N} m\tilde{u}_k^2 = \sum_{\alpha,\beta=[a,c,h]} m C_{\alpha\beta} \alpha^3 \beta^3 / (h^4 E^2 F^2), \quad (14)$$

where:

$$
\begin{aligned}
&C_{aa} = (751n^2 - 792n + 216)/36,\\
&C_{cc} = (n^2 + 216)/36,\\
&C_{hh} = (270n + 1873)/9,\\
&C_{ac} = -(5n^2 - 396n + 216)/18,\\
&C_{ah} = (793n - 468)/9,\\
&C_{ch} = (13n + 468)/9.
\end{aligned} \tag{15}
$$

Thus, the upper estimate of the first frequency of the truss, depending on the number of panels, can be obtained by the formula:

$$
\omega_R = h\sqrt{\frac{EF\sum\limits_{\alpha=[a,c,h]} C_\alpha \alpha^3}{m\sum\limits_{\alpha,\beta=[a,c,h]} C_{\alpha\beta}\alpha^3\beta^3}} \tag{16}
$$

with coefficients (13), (15) depending only on the construction order $n$.

### 3.3.2 Dunkerley score

The lower estimate of the first oscillation frequency is obtained by the Dunkerley formula:

$$
\omega_D^{-2} = \sum_{i=1}^{K} \omega_i^{-2}\ , \tag{17}
$$

where $\omega_i$ is the oscillation frequency of one mass $m$ located at node *i*. To calculate partial frequencies $\omega_i$, we compose equation (4) in the scalar form:

$$
m\ddot{z}_i + D_i z_i = 0,
$$

where $D_i$ is the scalar stiffness coefficient (i is the mass number). Load oscillation frequency

$$
\omega_i = \sqrt{D_i / m}. \tag{18}
$$

The stiffness coefficient, the reciprocal of the compliance coefficient, is determined by the Maxwell-Mohr formula (1):

$$
\delta_i = 1/D_i = \sum_{\alpha=1}^{n_s}\left(\tilde{S}_\alpha^{(i)}\right)^2 l_\alpha/(EF).
$$

From (17) and (18) follows $\omega_D^{-2} = \sum\limits_{i=1}^{K}\omega_i^{-2} = m\sum\limits_{i=1}^{K}\frac{1}{D_i}$. Since compliance is the reciprocal of stiffness $1/D_i = \delta_i$, then:

$$
\omega_D^{-2} = m\sum_{i=1}^{K}\delta_i = m\sum_{i=1}^{K}\sum_{\alpha=1}^{n_s}\left(\tilde{S}_\alpha^{(i)}\right)^2 l_\alpha/(EF) = m\Sigma_n/(h^2EF). \tag{19}
$$

Let us successively calculate the sums $\Sigma_n = h^2\sum\limits_{i=1}^{K}\sum\limits_{\alpha=1}^{n_s}\left(\tilde{S}_\alpha^{(i)}\right)^2 l_\alpha$ при $n$ = 2,3,4,...

$$
\begin{aligned}
&\Sigma_2 = \frac{47a^3 + 107c^3 + 293h^3}{6},\\
&\Sigma_3 = \frac{279a^3 + 243c^3 + 926h^3}{18},\\
&\Sigma_4 = \frac{562a^3 + 280c^3 + 1583h^3}{24},\\
&\Sigma_5 = \frac{4715a^3 + 1607c^3 + 12482h^3}{150},\ldots
\end{aligned}
$$

Let us calculate the common terms of the sequences of coefficients in these expressions. We get $\Sigma_n = \sum_{\alpha=[a,c,d,h]} r_\alpha \alpha^3$, where:

$$r_a = (49n^2 - 60n + 18)/(6n),$$
$$r_c = (n^3 + 36n^2 + 102n + 72)/(6n^2), \quad (20)$$
$$r_h = (54n^3 + n^2 - 228n + 606)/(3n^2).$$

When deriving expressions for the coefficients (20), the rgf_findrecur operator was used to compose the recursive equations of the Maple system. The resolve operator was used to solve these equations and obtain the common terms of the sequences. The direct application of such an algorithm does not give a result in this case, since the members of the sequences have the form of fractions, in which not only numerators, but denominators depend on *n*. The Maple system operators are not designed to define the common members of such sequences. Success was achieved only because it was possible to guess the type of denominators. An alternative method for finding common members of sequences is provided by the FindSequenceFunction operator from the Wolfram Mathematica symbolic mathematics package. As a result, the lower estimate for the first frequency according to Dunkerley is:

$$\omega_D = h\sqrt{\frac{EF}{m \sum_{\alpha=[a,c,d,h]} r_\alpha \alpha^3}}. \quad (21)$$

Formula (21) almost coincides in form with formula (16) obtained by the Rayleigh method, but Dunkerley's estimate (20) is much simpler. In this formula, the desired coefficients are contained only in the denominator.

### 3.3.3 Numerical example

To estimate the error of the estimates found, consider a truss with *n* panels with dimensions $h = 1\text{m}$, $a = 4\text{ m}$. Mass of cargo $m = 400\,\text{kg}$. We take the rigidity of the steel rods of the truss N. Figure (6) plots the dependences on the number of panels of the upper estimate of the lowest frequency $\omega_R$ according to formula (17), the Dunkerley estimate $\omega_D$ (21), and the numerical value of the first frequency of the spectrum found by analyzing the solution to the problem of oscillation of a system with *K* degrees of freedom.

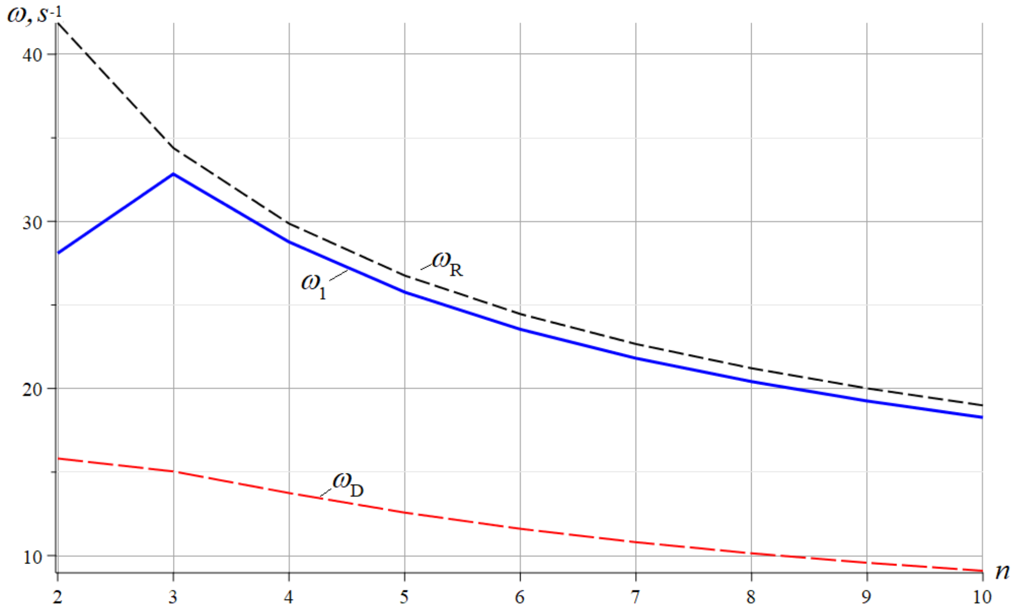


**Fig. 6 - Dependence on the number of panels of the first oscillation frequency $\omega_R$ according to the Rayleigh method, the frequency $\omega_D$ according to the Dunkerley method, and the first frequency $\omega_1$ of the spectrum obtained numerically**

As the number of panels increases, the frequency decreases. It is also obvious that the error of the Dunkerley method is much larger than that of the Rayleigh method. For a more accurate estimate of the error, we introduce the relative errors $\varepsilon_D = |\omega_D - \omega_1|/\omega_1$, $\varepsilon_R = |\omega_R - \omega_1|/\omega_1$. Graph 7 shows that the accuracy of estimates depends not so much on the number of panels as on the height h. For low coverage heights, the error of the Rayleigh method does not exceed 5%. For high altitudes, the accuracy of the Dunkerley method is unsatisfactory, almost does not depend on the number of panels, and the error is more than 60%.

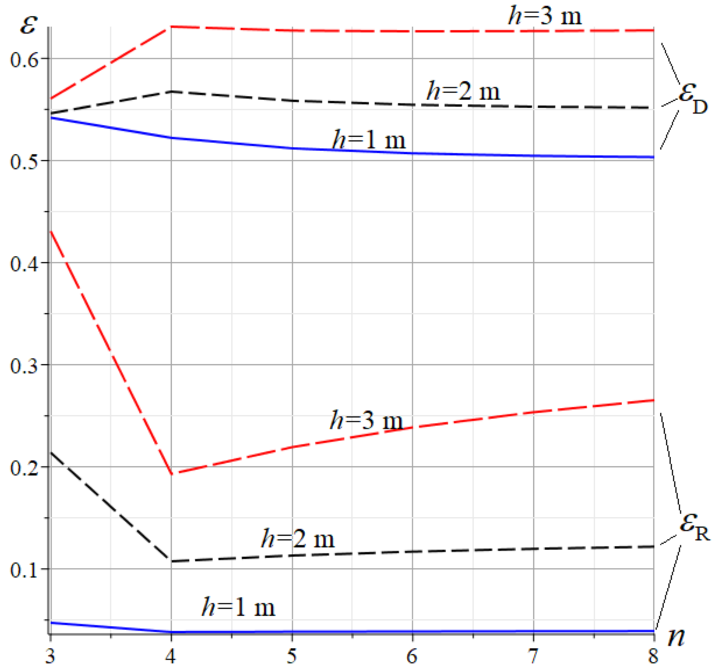


Fig. 7 - Relative error of the upper and lower estimates

## 4 Conclusion

A new scheme of a statically determined truss of spatial coverage is proposed, for which formulas for deflections and estimates of the first natural frequency for an arbitrary number of panels are derived by induction. The resulting calculation formulas for deflections within the framework of the adopted model were obtained without any simplifying assumptions and can be used both to assess the accuracy of numerical solutions and to preliminary assess the operational characteristics of the structure being designed. These formulas are especially effective as an alternative to numerical calculations, especially for structures of a high order of regularity, which are characterized by the inevitable accumulation of rounding errors and large expenditures of computer time.

The found analytical estimates of the lowest oscillation frequency showed that the accuracy of the Rayleigh estimate is quite sufficient for using the obtained formula in calculations, and the accuracy of the simpler lower estimate of the Dunkerley frequency is unsatisfactory, although, judging by graph 5, the nature of the dependence of this value on the number of panels is similar to within factor to the numerical solution.

Analytical solutions make it possible to analyze and select the most optimal parameters of a simplified model of a structure being designed without using a numerical solution of a real structure. One of the advantages of the analytical solution is the independence of its accuracy from the order of regularity of the construction. In three-dimensional problems requiring a large number of calculations, this advantage is more pronounced than for planar systems.

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